AD-A050 043 FLORIDA UNIV GAINESVILLE DEPT OF INDUSTRIAL AND SYS--ETC F/G 13/8
THE REVERSIBILITY PROPERTY OF PRODUCTION LINES.(U)
JAN 78 E J MUTH DAHC04-75-G-0150
UNCLASSIFIED RR-78-1 ARO-12640.23-M NL

1 OF 1
AD
A050043

END
DATE
FILMED
3-78
DDC

ARO 12640.23-M

12

AD A050043

RESEARCH REPORT

THE REVERSIBILITY PROPERTY OF PRODUCTION LINES

Research Report No. 78-1

by

Eginhard J. Muth

January, 1978

Department of Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611

This research was supported in part by the Army Research Office, Triangle Park, NC, under contract number DAHC04-75-G-0150.

THE FINDINGS OF THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION, UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
78-1

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
6 The Reversibili[illegible]operty of Production Lines,

5. TYPE OF REPORT & PERIOD COVERED
Technical

6. PERFORMING ORG. REPORT NUMBER
14 RR-78-1

7. AUTHOR(s)
10 Eginhard J./Muth

8. CONTRACT OR GRANT NUMBER(s)
15 DAHC04-75-G-0150

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611

12 20p.

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
16 20061102A14D Rsch in & Appl of Applied Math.

11. CONTROLLING OFFICE NAME AND ADDRESS
U.S. Army Research Office
P.O. Box 12211
Triangle Park, NC 27709

12. REPORT DATE
11 Jan[redacted] 1978

13. NUMBER OF PAGES
19

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
18 ARO 19 12640.23-M

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
N/A

16. DISTRIBUTION STATEMENT (of this Report)
9 Research rept,

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

N/A

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Production lines
Applied stochastic processes
Queues in tandem

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A production line is treated as a series arrangement of k work stations. An unlimited supply of raw production items is available at the first station, and each item passes through all of the stations in sequence. The service time for a single item at station j is assumed to be a random variable with a probability distribution peculiar to that station. In this mode of operation any station will at any time be either busy, or idle, or blocked. A measure of the productivity of such a line is its mean production rate r. → over

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

404 399

LB

20. (cont'd)

It has been conjectured that the production rate remains invariant under reversal of the production line. Line reversal means that every item passes through the stations in the reverse order, that is, beginning with station k and ending with station 1. A general proof of the reversibility property is given. First it is shown that with predetermined service times the total time required to process n dissimilar items through k dissimilar stations does not change when the order of the stations and the order of the items is reversed. Then it is shown for the stochastic case that the order of the items does not affect the production rate.

ACCESSION for		
NTIS	White Section	☑
DDC	Buff Section	☐
UNANNOUNCED		☐
JUSTIFICATION		
BY		
DISTRIBUTION/AVAILABILITY CODES		
Dist.	AVAIL. and/or SPECIAL	
A		

TABLE OF CONTENTS

ABSTRACT

A production line is treated as a series arrangement of k work stations. An unlimited supply of raw production items is available at the first station, and each item passes through all of the stations in sequence. The service time for a single item at station j is assumed to be a random variable with a probability distribution peculiar to that station. In this mode of operation any station will at any time be either busy, or idle, or blocked. A measure of the productivity of such a line is its mean production rate r. It has been conjectured that the production rate remains invariant under reversal of the production line. Line reversal means that every item passes through the stations in the reverse order, that is, beginning with station k and ending with station 1. A general proof of the reversibility property is given. First it is shown that with predetermined service times the total time required to process n dissimilar items through k dissimilar stations does not change when the order of the stations and the order of the items is reversed. Then it is shown for the stochastic case that the order of the items does not affect the production rate.

INTRODUCTION AND DEFINITIONS

The production line has been the object of a good deal of study in the past. Much of the past work has involved the development of analytical solutions or empirical formulas for the production rate. Another aspect of production lines that has received attention is the optimization of production rate. An important question in this context is how the production rate is affected by a rearrangement of the order of the servers. A special case of rearrangement is that of line reversal, to be defined in the following section. It has been conjectured by Hillier and Boling [1] and Knott [3], that the production rate does not change under line reversal. Such a conjecture is strongly motivated by a few special cases for which it is known to be true and by simulation results for more general cases. However, a proof of the reversibility property has not been published. To give such a proof is the purpose of this paper. A less complete version of this proof was presented informally at an ORSA meeting, see [6].

We introduce the problem by characterizing the production line shown in Figure 1a. The line consists of k dissimilar work stations (servers) labeled 1, 2, ..., k, and arranged in series in that order. There is an unlimited supply of identical production items (customers) at station 1. Each raw production item enters the line at station 1, passes through all stations in order, and leaves station k as a finished item. The service time of any item at station j is a nonnegative random variable denoted S_j. The random variables $S_1, S_2, \ldots, S_k$ are statistically independent, their distributions are arbitrary and not identical in general. The service time of item i at station j is S_{ij}, and the sequence S_{ij}, $i = 1, 2, \ldots$, is identically and

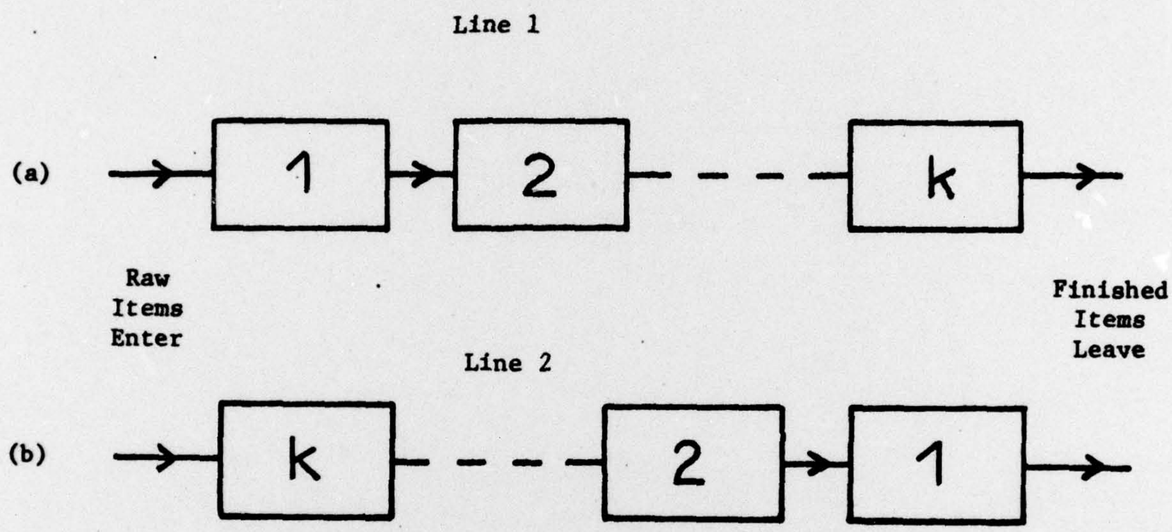

Figure 1. The k-station production line.

independently distributed. It is assumed that a station cannot break down and that each station can service only one item at a time. However, the case of station breakdown, under certain assumptions, can be reduced to the case treated here, whereby periods of station breakdown become a part of the service time. Every station is at any time in one of three possible states. A station is busy when it is servicing an item, blocked when it has completed service but cannot pass on its item to the next station because that station is still busy or blocked, and idle when it is neither busy nor blocked. Idleness is caused when a station has passed on its item to the next station and has not yet received a new item from the preceding station. Service begins immediately at the time at which an item arrives at an idle station. The first station can never be idle, similarly the last station can never be blocked. There may be buffer spaces (in-process storage) provided between stations to diminish the occurrence of blocking. This case does not require special treatment since a buffer which can hold ℓ items is equivalent to ℓ stations in series, where each station has zero service time, see [5]. We note that the phenomena of blocking and idling introduce an element which is not found in ordinary queues; this makes analysis of production lines more difficult.

One important measure of the efficiency of a line is the mean production rate r. It is commonly defined to be the expected number of items, per unit time, released from the last station in the long run. Let N(t) be the number of items released from station k in the time interval (0, t). Then we have

$$r = \lim_{t \to \infty} \frac{E[N(t)]}{t} \tag{1}$$

where $E[\cdot]$ denotes the expectation operator. An alternative and equivalent

definition is

$$r = 1/E[D] \tag{2}$$

where D may be either the interdemand time at station 1 or the interdeparture time at station k*. For the purpose of our subsequent analysis and proof we will define r in terms of the production time P(n) of the first n items to pass through the line, that is, the time which elapses between the arrival of item 1 at station 1 and the departure of item n from station k. We have

$$r = \lim_{n\to\infty} \frac{n}{E[P(n)]} \tag{3}$$

Clearly, (2) is obtained from (3) by putting $E[P(n)] = E[P(1)] + (n-1)E[D]$.

The mean production rate is generally a function of the distribution of the service time of each of the k servers and the order of the servers. Analytical closed form expressions for this functional dependence have been obtained only for very simple cases, where $k \leq 3$. For example, for the case $k = 3$ with exponentially distributed service times having mean values $1/\mu_1$, $1/\mu_2$ and $1/\mu_3$, Hunt [2] obtained an expression for r, his equations (21), (22), (23), which is the ratio of two polynomials in μ_1, μ_2, μ_3. The numerator polynomial is of degree 6 and contains 22 terms, the denominator polynomial is of degree 7 and contains 24 terms. The expression is already so complicated that it does not let one obtain much insight in the behavior of the function. Another closed form expression is given in [7], equation (102), for $k = 3$ and uniformly distributed service times, but only for the case where all three servers are identical. These two examples underline the difficulty of the problem of obtaining analytical results, and explain why most investigations of production lines rely on simulation or on numerical methods.

*It could, in fact, be the interdemand time or the interdeparture time at any station.

PRODUCTION LINE REVERSAL

The production line defined in the preceding section is <u>reversed</u> when each item passes through the work stations in the order k, k - 1, ..., 2, 1, that is, the item enters the line at station k and departs from it at station 1. Figure 1b shows the reversed line. Let r_1 be the mean production rate of the original line and r_2 be that of the reversed line. It has been conjectured that $r_1 = r_2$. This property will be referred to as <u>reversibility</u>. The conjecture that reversibility holds under the most general conditions is supported by the following special cases.

(i) Reversibility holds for k = 2 and general service time distributions, in which case we have [5]

$$r = \frac{1}{E[\max(S_1, S_2)]} \tag{4}$$

(ii) Reversibility holds for k = 3 when the service times are exponentially distributed as shown in [4] and as can be deduced from the aforementioned expression by Hunt [2] which is symmetric in μ_1 and μ_3. That expression also shows that the production rate does change when servers are arranged in the order 1, 3, 2 or 3, 1, 2.

(iii) Reversibility holds for any k when the service times are fixed (deterministic), see [5]. In that case we have

$$r = \frac{1}{\max(S_1, S_2, \ldots, S_k)} \tag{5}$$

and the production rate does not at all depend on the order of the servers.

In the following we present a proof of reversibility which holds for any k and for arbitrarily distributed service times. It should be apparent

from the discussion of the foregoing section that it will not be possible to find a general proof which relies on a closed form expression for the production rate. Our proof uses implicitly the description of a production line by the sequence of <u>holding times</u> (service times plus blocking times) of each item. The corresponding stochastic process was first applied in [5] and was formalized and called the <u>holding time process</u> in [7].

The key idea, which led to the development of the proof to follow, is that of a movie run backwards. Suppose that a movie is taken of a production line, showing every station and the items being worked on, over the period of time required to process n items. When the movie is run backwards, items will proceed through the line in reverse, and although the mode of operation is not the same as for the original line (blocking periods now precede service periods), there is no contradiction; the service times are the same as in the forward case, and it takes exactly the same total time to process n items. It is easy to reason that this total time cannot be reduced by letting the backward case conform to the normal mode of operation.

An important insight gained from the Gedankenexperiment with the movie in reverse is that the items need to proceed through the reversed line in reverse order if the production times for the two cases are to be equal. This situation will be referred to as <u>line and time reversal</u>. Accordingly, our proof consists of two parts dealt with in two separate sections. In the first part, we show that for any deterministic sequence of service times the production time of n items is invariant under line and time reversal. In the second part we extend this result to the stochastic case where we show that the expected value of production time is invariant under line reversal alone.

THE DETERMINISTIC CASE

In this section we treat the case of n items whose service times at each of the k stations are predetermined. Thus we have a sequence $\{S_{ij}\}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$, in which the values S_{ij} are arbitrary but fixed. In this sense we are speaking of the deterministic case. However, we may think of the S_{ij} either as the known and fixed service times for n dissimilar items or as a particular realization of the random service times for n identical items. It should be noted in this connection that the so called machine scheduling problem seeks to find that order of n dissimilar items with fixed service times which minimizes the production time.

The production time of the n items is $P_1(n)$. It must be some function of all n x k values S_{ij}. We will develop an expression for $P_1(n)$ with the aid of the activity network shown in Figure 2. However, such an expression is not required, for it is the structure of the activity network which leads immediately to the conclusion that the production time is invariant under line and time reversal. An activity network is a directed graph which represents the time required to carry out a project, in our case the time required to process n items. The arcs of the network represent the times required to carry out certain activities and the nodes represent points in time at which certain events occur. In our network all nodes are AND nodes. This means that a node is realized only at the time at which all activities associated with arcs leading into that node are completed. All activities associated with arcs emanating from a node are begun at the time at which the node is realized. The horizontal arcs of the network represent the service times S_{ij} and the vertical arcs represent dummy activities of duration zero.

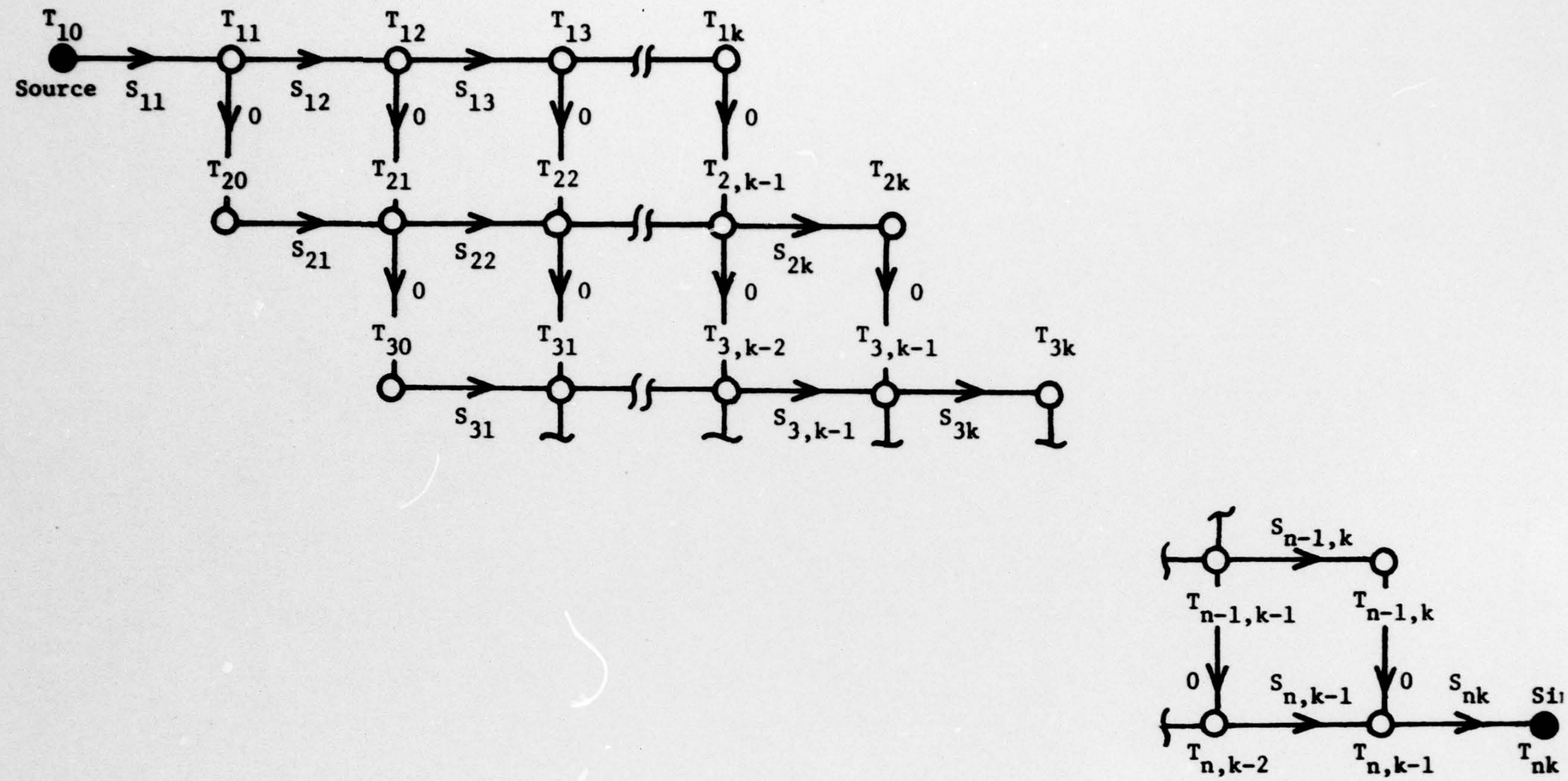

Figure 2. Activity network representing the time required to process n items through k stations.

The arc labeled S_{ij} leads into the node ij, which represents T_{ij}, the time at which item i departs from station j. T_{i0} is the time at which item i enters station 1. The vertical arcs provide the coupling between the passage of two successive items through the line; they thus account for the blocking phenomenon. For example, item 2 cannot move into station 2 before item 1 has cleared station 2, that is, before the node 12 has been realized. This is assured by the vertical arc from node 12 to node 21. We observe that nodes lying on a vertical line have the same index sum. The departure times T_{ij} and $T_{i-1,\ j+1}$ are represented by adjacent nodes on the same vertical line, see Figure 3. It is apparent that $T_{ij} \geq T_{i-1,\ j+1}$. More specifically we have the relation

$$T_{ij} = \max\{T_{i-1,\ j+1}, (T_{i,\ j-1} + S_{ij})\}\ ,\ i > 1,\ j = 1,2,\ldots,k-1 \qquad (6)$$

By using this relation, it now transpires that every T_{ij} is expressible as the maximum of certain sums of service times. For example we have

$$T_{11} = S_{11}$$

$$T_{21} = \max\{(S_{11} + S_{12}), (S_{11} + S_{21})\}$$

$$T_{22} = \max\{(S_{11} + S_{12} + S_{13}), (S_{11} + S_{12} + S_{22}), (S_{11} + S_{21} + S_{22})\}$$

In the expression for T_{22} the sum $(S_{11} + S_{12} + S_{13})$ represents a path from the source node 10 to the node 22. There are three possible paths between these two nodes and the node realization time T_{22} is equal to the longest path. We now define a path from the source node to node ij as the sum

$$R_{ij}(L, M) = \sum_{k=2}^{i+j} S_{\ell(k),\ m(k)} \qquad (7)$$

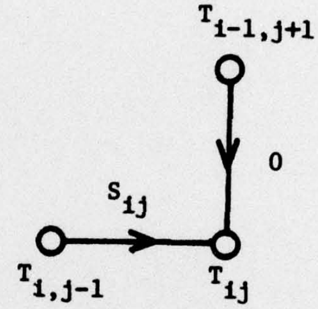

Figure 3. Illustration of Equation (6).

where

$$L = (\ell(2),\ \ell(3),\ \dots,\ \ell(i+j))$$

$$M = (m(2),\ m(3),\ \dots,\ m(i+j)) \tag{8}$$

and where the progression of the indices $\ell(k)$ and $m(k)$ satisfies the constraints

$$\ell(2) = m(2) = 1$$

$$\ell(k) + m(k) = k$$

$$\ell(k) \leq \ell(k+1) \leq \ell(k)+1 \tag{9}$$

It then follows that

$$T_{ij} = \max_{L,M} R_{ij}(L,\ M) \tag{10}$$

Also, the production time $P_1(n)$ corresponds to the longest path from the source node 10 to the sink node nk, that is

$$P_1(n) = T_{nk} = \max_{L,M} R_{nk}(L,\ M) \tag{11}$$

Now let $P_3(n)$ represent the production time for the same n items under line and time reversal. Clearly, the activity network for this case is obtained from the network of Figure 2 by reversing the direction of each arrow and by letting node nk be the source node and node 10 be the sink node. This implies that T_{ij} becomes the time at which item i enters station j. Since the longest path from node 10 to node nk is the same as the longest path from node nk to node 10 it follows that

$$P_1(n) = P_3(n) \tag{12}$$

and the proof of the first part is complete.

THE STOCHASTIC CASE

We let $M_1(n)$ be the matrix whose elements are the service times S_{ij}, that is

$$M_1(n) = \begin{bmatrix} S_{11} & \cdots & S_{1k} \\ \vdots & & \vdots \\ S_{n1} & \cdots & S_{nk} \end{bmatrix} \tag{13}$$

The matrix corresponding to line reversal alone is

$$M_2(n) = \begin{bmatrix} S_{1k} & \cdots & S_{11} \\ \vdots & & \vdots \\ S_{nk} & \cdots & S_{n1} \end{bmatrix} \tag{14}$$

and for line and time reversal we have

$$M_3(n) = \begin{bmatrix} S_{nk} & \cdots & S_{n1} \\ \vdots & & \vdots \\ S_{1k} & \cdots & S_{11} \end{bmatrix} \tag{15}$$

The production time as developed in (11) is a function $f\{\cdot\}$ of the matrix of service times, namely

$$P_i(n) = f\{M_i(n)\}\ , \qquad i = 1, 2, 3 \tag{16}$$

It was shown that $P_3(n) = P_1(n)$, but in general we have $P_2(n) \neq P_1(n)$.

In the stochastic case the service times S_{ij} as well as the production times $P_i(n)$ are random variables. Because of (16), the distribution function of $P_i(n)$

$$F_i(x) = P[P_i(n) \leq x] , \qquad i = 1, 2, 3 \tag{17}$$

is some function of the joint distribution of the elements of the random matrix $M_i(n)$. All the elements in any given column of $M_i(n)$ are identically and independently distributed random variables, thus the order of the elements in any column may be rearranged without changing the probability properties of $M_i(n)$. Since $M_2(n)$ is obtained by reversing all columns of $M_3(n)$ it follows that $M_2(n)$ and $M_3(n)$ are equivalent, in the sense that their probability measures are equal. Thus we obtain

$$F_3(x) = F_2(x) \tag{18}$$

A similar conclusion is that a server cannot distinguish one item from another because the sequence S_{ij}, $i = 1, 2, \ldots, n$, is identically and independently distributed. Hence we may define S_{ij} to be the service time at station j of the ith item which enters the station, instead of the service time of the item with label i. With this definition we have the stronger condition $M_3(n) = M_2(n)$, and both matrices are represented by (14). From (18) follows

$$E[P_3(n)] = E[P_2(n)] \tag{19}$$

and after substituting (12)

$$E[P_1(n)] = E[P_2(n)] \tag{20}$$

Finally, applying (20) to (3) gives

$$r_1 = r_2 \tag{21}$$

and the proof is complete.

REFERENCES

[1] Hillier, Frederick S. and Boling, Ronald W., "The Effect of Some Design Factors on the Efficiency of Production Lines with Variable Operation Times," Journal of Industrial Engineering, Vol. XVII, (1966), p. 651.

[2] Hunt, G. C., "Sequential Arrays of Waiting Lines," Opns. Res., Vol. 4 (1956), p. 674.

[3] Knott, Allen D., "Aggregation Laws for a Series of Finite Queues," Unpublished Working Paper.

[4] Makino, T., "On the Mean Passage Time Concerning Some Queueing Problems of the Tandem Type," J. Opns. Res. Soc. Japan, Vol. 7, No. 1, (1974), p. 17.

[5] Muth, E. J., "The Production Rate of a Series of Work Stations with Variable Service Times," Int. J. Prod. Res., Vol. 11, No. 2, (1973), p. 155.

[6] Muth, E. J. and Mehta, P. D., "On the Reversibility of Production Lines," Bulletin of the Operations Research Society of America, Vol. 22, Supplement 1, Spring, (1974), p. B-118.

[7] Muth, E. J., "Numerical Methods Applicable to a Production Line with Stochastic Servers," in Algorithmic Methods in Probability, M. F. Neuts, Editor, TIMS Studies in the Management Sciences, Vol. 7 (1977) p. 143.